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## On Casimir's Ghost

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### Abstract

We define on the universal enveloping superalgebra of  $osp(1|2n)$  a nonstandard adjoint action, endowing it with a module structure. This allows, in particular, to construct a bosonic operator which anticommutes with all the fermionic generators and which appears to be the square root of a certain Casimir operator.

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# 1 On $osp(1|2n)$

It appeared recently [1] that the study of representations and centre of the quantum universal enveloping superalgebra  $\mathcal{U}_q(osp(1|2))$  was greatly simplified by the use of a particular operator called the Scasimir. This operator, first written in [2] is the  $q$ -deformation of a classical operator of  $\mathcal{U}(osp(1|2))$  introduced in [3, 4]. This operator has the following property: it anticommutes with the fermionic generators and commutes with the bosonic ones, although it has a bosonic character. Moreover, the Scasimir can be seen as the square root of the quadratic Casimir element.

The specificities of the superalgebras  $osp(1|2n)$  (compared to the other simple Lie superalgebras) has led us to think that such a scheme should also hold for  $osp(1|2n)$ . The aim of this work was originally to find the explicit expression of the Scasimir, if it exists. It appeared that the existence of a Scasimir is actually explained by a more general structure inherent to  $osp(1|2n)$ , built on a nonstandard adjoint action. This action endows  $\mathcal{U}(osp(1|2n))$  with a structure of  $osp(1|2n)$ -module, in which a one-dimensional submodule corresponds to the Scasimir.

The paper is organized as follows: the end of this section is devoted to some notations. In section 2, we define the nonstandard action. We also define a remarkable subspace  $\Omega \subset \mathcal{U}(osp(1|2n))$ . It is proved in section 3 that  $\Omega$  is left stable by this action and its decomposition into simple  $osp(1|2n)$ -submodules is accomplished. In section 4, we show that  $\mathcal{U}(osp(1|2n))$  itself is also a direct sum of finite dimensional  $osp(1|2n)$ -modules, which is explicitly given in the case of  $osp(1|2)$ . Finally, in section 5, we compute the expression of the Scasimir.

The basic Lie superalgebra  $osp(1|2n)$  is defined, in the Cartan–Weyl basis, by the generators  $\sigma_a$  and  $\sigma_{ab}$  for  $1 \leq a, b \leq 2n$  and the relations

$$\begin{aligned} \{\sigma_a, \sigma_b\} &= \sigma_{ab} \\ [\sigma_a, \sigma_{bc}] &= -g_{ab}\sigma_c - g_{ac}\sigma_b \\ [\sigma_{ab}, \sigma_{cd}] &= -g_{ac}\sigma_{bd} - g_{ad}\sigma_{bc} - g_{bc}\sigma_{ad} - g_{bd}\sigma_{ac} \end{aligned} \tag{1}$$

where the  $2n \times 2n$  matrix  $(g_{ab})$  is given by

$$(g_{ab}) = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \tag{2}$$

where  $I_n$  is the  $n \times n$  unit matrix. The bosonic (or even) generators  $\sigma_{ab} = \sigma_{ba}$  generate the  $sp(2n)$  subalgebra of  $osp(1|2n)$ , while the fermionic (or odd) generators  $\sigma_a$  form the fundamental representation of  $sp(2n)$ .

A Cartan subalgebra  $\mathcal{H}$  can be obtained by choosing  $H_a = \sum_b g^{ba}\sigma_{ab}$  where  $1 \leq a \leq n$ . In the following, we will denote by  $\alpha$  the indices in  $\{1, \dots, n\}$  and define  $\bar{\alpha} = \alpha + n$  so that  $\bar{\alpha} \in \{n+1, \dots, 2n\}$ . The fermionic generators  $\sigma_\alpha$  and  $\sigma_{\bar{\alpha}}$  will be called conjugated.

By convention  $\bar{\bar{\alpha}} = \alpha$ . The Cartan generators are then the  $\sigma_{\alpha\bar{\alpha}}$ . With respect to this Cartan subalgebra, the generators  $\sigma_\alpha$ , (respectively  $\sigma_{\bar{\alpha}}$ ), act as raising (respectively lowering) operators. A system of simple root generators can be chosen as  $\{\sigma_{1\bar{2}}, \sigma_{2\bar{3}}, \dots, \sigma_{n-1\bar{n}}, \sigma_n\}$ . Then the positive root generators are  $\sigma_\alpha$ ,  $\sigma_{\alpha\alpha}$ , and  $\sigma_{\alpha\beta}$ ,  $\sigma_{\alpha\bar{\beta}}$  for  $1 \leq \alpha < \beta \leq n$ .

## 2 On a nonstandard action

It is known that the standard adjoint action of the generators  $\sigma_{ab}$  and  $\sigma_a$  on an element  $x$  of  $\mathcal{U}(\mathfrak{osp}(1|2n))$  given by  $[\sigma_{ab}, x]$  and  $[\sigma_a, x]_\pm$  endows  $\mathcal{U}(\mathfrak{osp}(1|2n))$  with a structure of  $\mathfrak{osp}(1|2n)$ -module.

Besides this action, we can define the following alternative action

$$\begin{aligned} \sigma_{ab} \triangleright x &= [\sigma_{ab}, x] \\ \sigma_a \triangleright x &= \{\sigma_a, x\} && \text{for even } x \\ \sigma_a \triangleright x &= [\sigma_a, x] && \text{for odd } x \end{aligned} \quad (3)$$

Obviously, the action of the bosonic part  $\sigma_{ab}$  through  $\triangleright$  again endows  $\mathcal{U}(\mathfrak{osp}(1|2n))$  with a structure of  $\mathfrak{sp}(2n)$ -module, since

$$[\sigma_{ab}, \sigma_{cd}] \triangleright x = \sigma_{ab} \triangleright (\sigma_{cd} \triangleright x) - \sigma_{cd} \triangleright (\sigma_{ab} \triangleright x). \quad (4)$$

Moreover, we can check that (3) also defines a structure of  $\mathfrak{osp}(1|2n)$ -module because  $\forall x \in \mathcal{U}(\mathfrak{osp}(1|2n))$ ,

$$\begin{aligned} \{\sigma_a, \sigma_b\} \triangleright x &= \sigma_a \triangleright (\sigma_b \triangleright x) + \sigma_b \triangleright (\sigma_a \triangleright x) \\ [\sigma_a, \sigma_{bc}] \triangleright x &= \sigma_a \triangleright (\sigma_{bc} \triangleright x) - \sigma_{bc} \triangleright (\sigma_a \triangleright x). \end{aligned} \quad (5)$$

In the universal enveloping algebra  $\mathcal{U}(\mathfrak{osp}(1|2n))$ , let us define the following completely antisymmetric polynomials in the fermionic generators  $\sigma_a$ :

$$[a_1 a_2 \dots a_p] \equiv \sum_{s \in \mathfrak{S}_p} \varepsilon(s) \sigma_{a_{s(1)}} \sigma_{a_{s(2)}} \dots \sigma_{a_{s(p)}} \quad (6)$$

where  $\mathfrak{S}_p$  is the permutation group of  $p$  elements and  $\varepsilon(s)$  is the signature of the permutation  $s$ . Note that this expression vanishes when two indices  $a_i$  and  $a_j$  coincide and that consequently  $[a_1 a_2 \dots a_p] = 0$  for  $p > 2n$ . We will denote by  $\Omega$  the vector subspace of  $\mathcal{U}(\mathfrak{osp}(1|2n))$  spanned by the  $[a_1 a_2 \dots a_p]$  for  $p \in \{0, \dots, 2n\}$ , with the convention that for  $p = 0$ , the value of  $[...]$  is 1.

**Proposition 1** *We have the decomposition*

$$\mathcal{U}(\mathfrak{osp}(1|2n)) = \mathcal{U}(\mathfrak{sp}(2n)) \otimes \Omega. \quad (7)$$

**Proof:** A Poincaré–Birkhoff–Witt basis  $\mathfrak{B}$  of  $\mathcal{U}(\mathfrak{osp}(1|2n))$  is given by the tensor product of a Poincaré–Birkhoff–Witt basis of  $\mathcal{U}(\mathfrak{sp}(2n))$  by elements of the form  $\sigma_{a_1} \dots \sigma_{a_p}$  with  $1 \leq a_1 < \dots < a_p \leq 2n$ . Considering an element  $\omega \in \Omega$ , it can be re-expressed in  $\mathfrak{B}$  using the commutation relations (1). The so-obtained result contains *i*) a product of fermions of the same degree as  $\omega$  and *ii*) a finite linear combination of elements of  $\mathfrak{B}$  with a fermionic part of strictly lower degree. It follows that there exists an invertible triangular matrix relating the basis  $\mathfrak{B}$  and the tensor product of the Poincaré–Birkhoff–Witt basis of  $\mathcal{U}(\mathfrak{sp}(2n))$  by the basis of  $\Omega$  given by the elements of the form (6). ■

### 3 On a nice subrepresentation

We will now consider the action of  $\mathfrak{osp}(1|2n)$  by  $\triangleright$  on  $\Omega$ .

**Proposition 2** *The vector space  $\Omega$  is stable under the action of any  $\sigma_{ab}$  by  $\triangleright$ , i.e.*

$$\sigma_{ab} \triangleright \Omega \subset \Omega. \quad (8)$$

*Then, under this action,  $\Omega$  is a  $\mathfrak{sp}(2n)$ -submodule of  $\mathcal{U}(\mathfrak{osp}(1|2n))$ . Moreover,  $\mathcal{U}(\mathfrak{osp}(1|2n))$  and  $\mathcal{U}(\mathfrak{sp}(2n)) \otimes \Omega$  are equivalent as  $\mathfrak{sp}(2n)$ -modules.*

**Proof:** The commutation relations (1) and the definition (3) lead to the formula

$$\sigma_{bc} \triangleright [a_1 a_2 \dots a_p] = - \sum_{i=1}^p g_{ba_i} [a_1 \dots \overset{(i)}{c} \dots a_p] - \sum_{i=1}^p g_{ca_i} [a_1 \dots \overset{(i)}{b} \dots a_p] \in \Omega \quad (9)$$

in which the notation  $\overset{(i)}{x}$  means that  $a_i$  is replaced by  $x$ . ■

**Proposition 3** *The vector space  $\Omega$  is stable under the action of any  $\sigma_a$  by  $\triangleright$ , i.e.*

$$\sigma_a \triangleright \Omega \subset \Omega. \quad (10)$$

*Then, the action of  $\triangleright$  endows  $\Omega$  with a structure of  $\mathfrak{osp}(1|2n)$ -submodule of  $\mathcal{U}(\mathfrak{osp}(1|2n))$ .*

**Proof:** The stability statement is proved by a direct calculation. We consider four cases, depending on whether  $a, \bar{a}$  belong to  $\{a_1, \dots, a_p\}$  or not.

Let us first take the case where  $a, \bar{a} \notin \{a_1, \dots, a_p\}$ . To this aim, we consider the polynomial  $[aa_1 a_2 \dots a_p]$  of  $\Omega$ , in which we move the  $\sigma_a$  generator to the left using the commutation

relations (1). We get

$$\begin{aligned}
[aa_1a_2 \dots a_p] &= \sigma_a \sum_{2k \leq p} \sum_{a'_1, \dots, a'_{p-2k}} [a'_1 \dots a'_{p-2k}] 2^k k! \binom{p+1}{2k+1} \prod_{i=1}^k (-g_{..}) \\
&\quad - \sum_{j=1}^p \sigma_{aa_j} \sum_{2k \leq p-1} \sum_{a'_1, \dots, a'_{p-2k-1}} [a'_1 \dots a'_{p-2k-1}] 2^k k! \binom{p+1}{2k+2} \prod_{i=1}^k (-g_{..})
\end{aligned} \tag{11}$$

$k$  counts for the number of pairs of conjugated fermionic generators missing in the right hand side, each pair  $\sigma_b, \sigma_{\bar{b}}$  being replaced by a factor  $-g_{b\bar{b}}$ . The remaining indices  $a'_1, a'_2, \dots$  shall appear in the same order as in the left hand side.

We repeat the procedure, starting again from  $[aa_1a_2 \dots a_p]$ , by moving the  $\sigma_a$  generator to the right. We then symmetrize or antisymmetrize according to the parity of  $p$ . One obtains

$$\begin{aligned}
2[aa_1a_2 \dots a_p] &= \sum_{2k \leq p} \sum_{a'_1, \dots, a'_{p-2k}} \sigma_a \triangleright [a'_1 \dots a'_{p-2k}] 2^k k! \binom{p+1}{2k+1} \prod_{i=1}^k (-g_{..}) \\
&\quad - \sum_{j=1}^p \sum_{2k \leq p-1} \sum_{a'_1, \dots, a'_{p-2k-1}} \sigma_{aa_j} \triangleright [a'_1 \dots a'_{p-2k-1}] 2^k k! \binom{p+1}{2k+2} \prod_{i=1}^k (-g_{..})
\end{aligned} \tag{12}$$

Notice that the first term of the right hand side with  $k = 0$  reduces to  $\sigma_a \triangleright [a_1a_2 \dots a_p]$ . It follows that  $\sigma_a \triangleright [a_1a_2 \dots a_p]$  can be expressed in terms of the left hand side and lower order terms of the right hand side (by virtue of Proposition 2, the second sum is an element of  $\Omega$ ). Then, by recursion on  $p$ ,  $\sigma_a \triangleright [a_1a_2 \dots a_p] \in \Omega$ .

We now consider the case when  $a$ , but not  $\bar{a}$ , belongs to the set  $\{a_1, \dots, a_p\}$ . Taking for instance  $a = a_1$ , we have

$$\sigma_a \triangleright [aa_2 \dots a_p] = 0 \tag{13}$$

The case when  $\bar{a}$ , but not  $a$ , belongs to the set  $\{a_1, \dots, a_p\}$  can be treated in an analogous way as the first one. One finds, taking for example  $\bar{a} = a_1$ ,

$$\begin{aligned}
2[a\bar{a}a_2 \dots a_p] &= \sum_{2k \leq p-1} \sum_{a'_1, \dots, a'_{p-2k}} \sigma_a \triangleright [a'_1 \dots a'_{p-2k}] 2^k k! \binom{p+1}{2k+1} \prod_{i=1}^k (-g_{..}) \\
&\quad + \sum_{j=2}^p \sum_{2k \leq p-2} \sum_{a'_1, \dots, a'_{p-2k-1}} \sigma_{aa_j} \triangleright [a'_1 \dots a'_{p-2k-1}] 2^k k! \binom{p+1}{2k+2} \prod_{i=1}^k (-g_{..}) \\
&\quad + 2g_{a\bar{a}} \sum_{2k \leq p-2} \sum_{a'_1, \dots, a'_{p-2k-1}} [a'_1 \dots a'_{p-2k-1}] 2^k k! \binom{p+1}{2k+3} \prod_{i=1}^k (-g_{..}) .
\end{aligned} \tag{14}$$

As before, this allows us to express  $\sigma_a \triangleright [\bar{a}a_2 \dots a_p]$  as a combination of elements of  $\Omega$  and lower order terms. By recursion on  $p$ , this proves that  $\sigma_a \triangleright [\bar{a}a_2 \dots a_p] \in \Omega$ .

The last case is when  $a, \bar{a} \in \{a_1, \dots, a_p\}$ . Taking  $a = a_1$  and  $\bar{a} = a_2$ , we now directly compute

$$\begin{aligned}
2\sigma_a \triangleright [a\bar{a}a_3 \dots a_p] = & \\
& -g_{a\bar{a}} \sum_{2k \leq p-2} \sum_{a'_1, \dots, a'_{p-2k-2}} \sigma_a \triangleright [a'_1 \dots a'_{p-2k-2}] 2^k k! (2k+1) \binom{p+1}{2k+3} \prod_{i=1}^k (-g_{..}) \\
& + g_{a\bar{a}} \sum_{j=3}^p \sum_{2k \leq p-3} \sum_{a'_1, \dots, a'_{p-2k-3}} \sigma_{aa_j} \triangleright [a'_1 \dots a'_{p-2k-3}] 2^{k+1} (k+1)! \binom{p+1}{2k+4} \prod_{i=1}^k (-g_{..})
\end{aligned} \tag{15}$$

■

**Proposition 4** *As an  $osp(1|2n)$ -module,  $\Omega$  is equivalent to the direct sum of the representations  $\mathcal{V}_j$  characterized by the highest weights  $(\underbrace{1, \dots, 1}_j, \underbrace{0, \dots, 0}_{n-j})$  with respect to  $\sigma_{1\bar{1}}, \dots, \sigma_{n\bar{n}}$ :*

$$\Omega \simeq \bigoplus_{j=0}^n \mathcal{V}_j \tag{16}$$

Notice that  $\mathcal{V}_0$  and  $\mathcal{V}_1$  are the trivial and fundamental representations of  $osp(1|2n)$ .

**Proof:** Let us recall that all the finite dimensional representations of  $osp(1|2n)$  are completely reducible [5, 6]. The representation  $\Omega$  thus decomposes into a sum of irreducible highest weight representations. What we need is to recognize in  $\Omega$  the highest weights of the representations  $\mathcal{V}_j$ , which are irreducible. We will show that the highest weight vectors have the form

$$v_j = [1 \ 2 \ \dots \ j \ j+1 \ \overline{j+1} \ \dots \ n \ \bar{n}] + \text{lower order terms} \tag{17}$$

where all the indices but  $\bar{1}, \dots, \bar{j}$  are present in the first term. The lower order terms are obtained from the first one by deleting pairs of conjugated indices  $\alpha, \bar{\alpha}$ . These highest weight vectors  $v_j$  have highest weight

$$(\underbrace{1, \dots, 1}_j, \underbrace{0, \dots, 0}_{n-j}) \tag{18}$$

with respect to  $\sigma_{1\bar{1}}, \dots, \sigma_{n\bar{n}}$ . We first note that  $\sigma_{\alpha \overline{\alpha+1}} \triangleright v_j = 0$  as long as the lower order terms in  $v_j$  are symmetric in the exchange of pairs of indices  $\{\alpha, \bar{\alpha}\}$  and  $\{\alpha+1, \overline{\alpha+1}\}$  for  $j+1 \leq \alpha \leq n-1$ . We now need to construct  $v_j$ , with this specification, such that  $\sigma_n \triangleright v_j = 0$ .

Let us first consider the action of  $\sigma_n$  on the leading term  $[1 \ 2 \ \dots \ j \ j+1 \ \overline{j+1} \ \dots \ n \ \bar{n}]$ . From (15) and repeated uses of (9) and (12), one obtains

$$\sigma_n \triangleright [1 \ 2 \ \dots \ j \ j+1 \ \overline{j+1} \ \dots \ n \ \bar{n}] = \sigma_n \triangleright \omega \quad (19)$$

where  $\omega \in \Omega$  is a linear combination of terms obtained from the leading one by deleting the pair  $n, \bar{n}$  and possibly other pairs of conjugated indices. By construction,  $\omega$  is invariant under the exchange of pairs of indices  $\{\alpha, \bar{\alpha}\}$  and  $\{\beta, \bar{\beta}\}$  for  $j+1 \leq \alpha, \beta \leq n-1$ . Let  $\tilde{\omega}$  be the expression obtained from  $\omega$  by extending this symmetry to the pair  $\{n, \bar{n}\}$ . One can remark that *i)* the degree of  $\omega$  and  $\tilde{\omega}$  is strictly less than the degree of the leading term and *ii)* all the terms of  $\tilde{\omega} - \omega$  contain the pair of indices  $\{n, \bar{n}\}$ . It follows that we can compute  $\sigma_n \triangleright (\tilde{\omega} - \omega)$  as we computed the action of  $\sigma_n$  on the leading term, and we get

$$\sigma_n \triangleright ([1 \ 2 \ \dots \ j \ j+1 \ \overline{j+1} \ \dots \ n \ \bar{n}] - \tilde{\omega}) = \sigma_n \triangleright \omega' \quad (20)$$

where the degree of  $\omega'$  is strictly less than the degree of  $\omega$ . Then, repeating the whole procedure as many times as necessary, we finally obtain  $\sigma_n \triangleright v_j = 0$ , where the element  $v_j$  is given by (17), the lower order terms being completely symmetric under the exchange of pairs of indices  $\{\alpha, \bar{\alpha}\}$  and  $\{\beta, \bar{\beta}\}$  for  $j+1 \leq \alpha, \beta \leq n$ . Since  $v_j$  is annihilated by all the simple positive root generators of  $osp(1|2n)$ , it is a highest weight vector.

The representation generated by the highest weight vector  $v_j$  is  $\mathcal{V}_j$ , of dimension  $\binom{2n+1}{j}$ . The sum of these dimensions is  $2^{2n} = \dim \Omega$  which concludes the proof. ■

The representations  $\mathcal{V}_j$  involved in this decomposition actually correspond [7] to the  $so(2n+1)$  representations obtained as the external powers  $\wedge^j \mathcal{V}$  of the fundamental representation  $\mathcal{V}$  of  $so(2n+1)$ .

**Corollary:** *There exists an element  $\mathcal{S}c$  of  $\mathcal{U}(osp(1|2n))$  belonging to  $\Omega$  that anticommutes with all the fermionic generators and commutes with all the bosonic ones. This element  $\mathcal{S}c$  is even with respect to the  $\mathbb{Z}_2$ -gradation of  $\mathcal{U}(osp(1|2n))$  and hence is not a Casimir operator. It is rather the square root of a central element. We call it the Scasimir. As in [3] for  $osp(1|2)$ , a normalized version of this operator would provide a realization of  $(-1)^H$ ,  $H = \sum_{a=1}^n H_a$  being the principal gradation or the “fermion number”.*

The proof of the corollary is obvious taking  $\mathcal{S}c = v_0$ , the highest weight vector of the trivial  $osp(1|2n)$ -representation of  $\Omega$ . The explicit expression of  $\mathcal{S}c$  is given in section 5.

## 4 On the decomposition of $\mathcal{U}(osp(1|2n))$

**Proposition 5** *As an  $osp(1|2n)$ -module (with the module structure endowed by the action of  $\triangleright$ ), the universal enveloping superalgebra  $\mathcal{U}(osp(1|2n))$  splits into finite dimensional representations.*

**Proof:**

Consider the natural filtration of  $\mathcal{U}(sp(2n))$

$$\mathbb{C} = \mathcal{U}_0 \subset \mathcal{U}_1 \subset \mathcal{U}_2 \subset \dots \subset \mathcal{U}_d \subset \dots \subset \mathcal{U}(sp(2n)) = \bigcup_{d=0}^{\infty} \mathcal{U}_d \quad (21)$$

allowing us to use the degree  $d$  of an element  $b \in \mathcal{U}(sp(2n))$  as the smallest  $d$  such that  $b \in \mathcal{U}_d$ .

Let us now prove that  $\mathcal{U}_d \otimes \Omega$  is stable under the action of  $\triangleright$ , which is enough to achieve the proof of the proposition. (Notice that this generalizes the proposition 3 which states the stability of  $\mathcal{U}_0 \otimes \Omega$ ). Taking an element  $b\omega \in \mathcal{U}(osp(1|2n)) = \mathcal{U}(sp(2n)) \otimes \Omega$  with  $b \in \mathcal{U}_d$  and  $\omega \in \Omega$ , one has for any fermionic generator  $\sigma \in osp(1|2n)$

$$\sigma \triangleright b\omega = [\sigma, b]\omega + b\sigma \triangleright \omega. \quad (22)$$

From Proposition 3, the term  $b\sigma \triangleright \omega$  belongs to  $\mathcal{U}_d \otimes \Omega$ . Using the commutation relations, the term  $[\sigma, b]$  can be expressed as a sum of terms of the type  $b_a \sigma_a$  with  $b_a \in \mathcal{U}_{d-1}$ . We are then led to check that  $\sigma_a \omega \in \mathcal{U}_1 \otimes \Omega$ . Using

$$\sigma_a \omega = \frac{1}{2}(\sigma_a \triangleright \omega + [\sigma_a, \omega]_{\pm}) \quad (23)$$

where  $[\cdot, \cdot]_{\pm}$  denotes the usual  $\mathbb{Z}_2$ -graded commutator, and equation (10), and writing  $\omega$  as in (6), one is left with

$$\begin{aligned} [\sigma_a, \omega]_{\pm} &= \sum_{s \in \mathfrak{S}_p} \varepsilon(s) \sum_{j=1}^p (-1)^{j-1} \sigma_{a_{s(1)}} \dots \sigma_{a_{s(j-1)}} \{ \sigma_a, \sigma_{a_{s(j)}} \} \sigma_{a_{s(j+1)}} \dots \sigma_{a_{s(p)}} \\ &= \sum_{s \in \mathfrak{S}_p} \varepsilon(s) \sum_{j=1}^p (-1)^{j-1} \left( \sigma_{aa_{s(j)}} \sigma_{a_{s(1)}} \dots \sigma_{a_{s(j-1)}} \sigma_{a_{s(j+1)}} \dots \sigma_{a_{s(p)}} \right. \\ &\quad \left. + \sigma_{a_{s(1)}} \dots \sigma_{a_{s(k-1)}} (g_{aa_{s(j-1)}} \sigma_{a_{s(j)}} + g_{a_{s(j)} a_{s(j-1)}} \sigma_a) \sigma_{a_{s(k)}} \dots \sigma_{a_{s(j-1)}} \sigma_{a_{s(j+1)}} \dots \sigma_{a_{s(p)}} \right) \end{aligned} \quad (24)$$

We use recursively the following procedure to the expression (24) : *i*) the fermionic generator  $\sigma_a$  is put in the first place, using the commutation relations (1), *ii*) the created bosonic generators are also put in the first place, *iii*) one uses iteratively equations (10), (23) and (24) on the generated terms of lower order. We eventually get a result that belongs to  $\mathcal{U}_1 \otimes \Omega$ . Therefore  $\sigma \triangleright b\omega \in \mathcal{U}_d \otimes \Omega$ , hence  $\mathcal{U}_d \otimes \Omega$  is stable under the action of  $\triangleright$ . ■

We now give the explicit decomposition in the case of  $osp(1|2)$ . We start with the decomposition of  $\mathcal{U}_d \otimes \Omega$  in terms of  $sp(2)$ -representations. The action  $\triangleright$  and the usual adjoint action of generators of  $sp(2)$  are indeed identical. Since  $\mathcal{U}_d \otimes \Omega$  is finite dimensional,



its decomposition in terms of  $osp(1|2)$ -representations is therefore uniquely determined. One finds

$$\mathcal{U}_d \otimes \Omega = \mathcal{R}_{d+\frac{1}{2}} \oplus \mathcal{R}_d \oplus \mathcal{R}_{d-\frac{1}{2}} \oplus \dots \oplus \mathcal{R}_0 \oplus C\mathcal{U}_{d-2} \otimes \Omega \quad (25)$$

where  $C$  is the quadratic Casimir of  $osp(1|2)$ . Hence, under the action of  $\triangleright$ ,  $\mathcal{U}(osp(1|2))$  decomposes into  $osp(1|2)$ -representation as follows:

$$\mathcal{U}(osp(1|2)) \simeq \mathcal{Z} \otimes \bigoplus_{j \in \mathbb{N}/2} \mathcal{R}_j \quad (26)$$

where  $\mathcal{Z}$  denotes the centre of  $\mathcal{U}(osp(1|2))$ , i.e. the algebra generated by 1 and  $C$ . This decomposition is actually the same as that obtained from the usual adjoint action [4].

We conjecture that this result generalizes for the case of  $osp(1|2n)$ , i.e. the decomposition of  $\mathcal{U}(osp(1|2n))$  into  $osp(1|2n)$ -representations is the same for the two actions (adjoint and  $\triangleright$ ). This is motivated by the following arguments: first we know that the decompositions in terms of  $sp(2n)$ -representations coincide (which would be sufficient for finite dimensional representations). Second the ambiguity in the gathering of  $sp(2n)$ -representations into  $osp(1|2n)$ -representations, which arises in the infinite dimensional case, can be lifted by looking at the finite dimensional subspaces  $\mathcal{U}_d \otimes \Omega$  for small  $d$ .

## 5 On the Scasimir operator

### 5.1 On its explicit expression

This section is devoted to the construction of  $\mathcal{S}c = v_0$ , the element of  $\Omega$  that anticommutes with all the fermionic generators. As stated in (17), we start from

$$v_0 = [1 \ \bar{1} \ 2 \ \bar{2} \ \dots \ n \ \bar{n}] + \text{lower order terms.} \quad (27)$$

We define for convenience

$$A_{2n-2k}^{(n)} \equiv \sum_{1 \leq \alpha_1 < \dots < \alpha_k \leq n} [1 \ \bar{1} \ \dots \ \widehat{\alpha_1} \ \widehat{\alpha_1} \ \dots \ \widehat{\alpha_k} \ \widehat{\alpha_k} \ \dots \ n \ \bar{n}] \quad (28)$$

in which the hat signals missing indices. For instance,  $A_0^{(n)} = 1$  and

$$\begin{aligned} A_2^{(1)} &= [1\bar{1}] \\ A_4^{(2)} &= [1\bar{1}2\bar{2}] \quad A_2^{(2)} = [1\bar{1}] + [2\bar{2}] \\ A_6^{(3)} &= [1\bar{1}2\bar{2}3\bar{3}] \quad A_4^{(3)} = [1\bar{1}2\bar{2}] + [1\bar{1}3\bar{3}] + [2\bar{2}3\bar{3}] \quad A_2^{(3)} = [1\bar{1}] + [2\bar{2}] + [3\bar{3}] \end{aligned} \quad (29)$$

The expressions  $A_{2n-2k}^{(n)}$  are invariant with respect to permutations of pairs of conjugated indices and they correspond, according to the previous section, to the terms involved in  $v_0$ :

$$v_0 = \sum_{k=0}^n x_k^{(n)} A_{2n-2k}^{(n)} \quad \text{with } x_0^{(n)} = 1. \quad (30)$$

From the equations used in Proposition 3, one can compute explicitly

$$\sigma_a \triangleright v_0 = \sigma_a C_a + \sum_b \sigma_{ab} C_{ab} \quad (31)$$

Now, demanding that the output is zero, one gets the two following recursion relations:

$$y_k^{(n)} = -\frac{1}{k} \sum_{p=0}^{k-1} y_p^{(n)} \frac{(2k+2)!/4}{(2p+2)!(2k-2p+1)!} \left( (2k-2p-1)(2n-2p+1) + 2p(2k-2p+1) \right) \quad (32)$$

$$y_k^{(n)} = -\frac{1}{k} \sum_{p=0}^{k-1} y_p^{(n)} \frac{(2k+2)!}{(2p+2)!(2k-2p+2)!} \left( (k-p)(2n-2p+1) + p(2k-2p+2) \right) \quad (33)$$

with

$$y_k^{(n)} = \frac{(2k+2)!(2n-2k)!}{2^k k! (2n+1)!} x_k^{(n)} \quad (34)$$

and the “initial condition”  $y_0^{(n)} = \frac{2}{2n+1}$ .

Consider the second recursion relation (33). It is solved using the generating function:

$$F_n(u) = \sum_{p=0}^{\infty} \frac{4^p y_p^{(n)}}{(2p+2)!} u^{2p} \quad (35)$$

Indeed, multiplying equation (33) by  $(2u)^{2k}$  and summing over  $k$ , one transforms the recursion relation into the following differential equation for the function  $F_n$ :

$$F_n'(u) = (2n+1) \left( \frac{1}{u} - \frac{\sinh(2u)}{\cosh(2u) - 1} \right) F_n(u) \quad (36)$$

whose solution is

$$F_n(u) = \frac{1}{2n+1} \left( \frac{u}{\sinh u} \right)^{2n+1} \quad (37)$$

so that

$$x_k^{(n)} = 2^{-k} k! \binom{2n}{2k} \frac{d^{2k}}{du^{2k}} \left( \frac{u}{\sinh u} \right)^{2n+1} \Big|_{u=0} \quad (38)$$

The same procedure applies for the first recursion relation (32). One finds exactly the same solution for the coefficients  $y_k^{(n)}$ . Of course, this is not surprising since it follows from the existence of  $v_0$  proved in the previous section, although the two recursion relations look different.

For illustration, we give hereafter the Scasimirs of  $\mathcal{U}(\text{osp}(1|2n))$ , for  $n = 1, \dots, 5$ :

$$\begin{aligned}
\mathcal{S}^{(1)} &= A_2^{(1)} - \frac{1}{2}A_0^{(1)} \\
&= [1\bar{1}] - \frac{1}{2} = \sigma_1\sigma_{\bar{1}} - \sigma_{\bar{1}}\sigma_1 - \frac{1}{2} \\
\mathcal{S}^{(2)} &= A_4^{(2)} - 5A_2^{(2)} + \frac{9}{2}A_0^{(2)} \\
&= [1\bar{1}2\bar{2}] - 5([1\bar{1}] + [2\bar{2}]) + \frac{9}{2} \\
\mathcal{S}^{(3)} &= A_6^{(3)} - \frac{35}{2}A_4^{(3)} + \frac{259}{2}A_2^{(3)} - \frac{675}{4}A_0^{(3)} \\
&= [1\bar{1}2\bar{2}3\bar{3}] - \frac{35}{2}([1\bar{1}2\bar{2}] + [1\bar{1}3\bar{3}] + [2\bar{2}3\bar{3}]) + \frac{259}{2}([1\bar{1}] + [2\bar{2}] + [3\bar{3}]) - \frac{675}{4} \\
\mathcal{S}^{(4)} &= A_8^{(4)} - 42A_6^{(4)} + 987A_4^{(4)} - 9687A_2^{(4)} + \frac{33075}{2}A_0^{(4)} \\
\mathcal{S}^{(5)} &= A_{10}^{(5)} - \frac{165}{2}A_8^{(5)} + 4389A_6^{(5)} - \frac{259215}{2}A_4^{(5)} + \frac{3171663}{2}A_2^{(5)} - \frac{13395375}{4}A_0^{(5)}
\end{aligned} \tag{39}$$

## 5.2 On a funny sum rule

The superalgebra  $\text{osp}(1|2n)$  can be realized in terms of bosonic oscillators. This realization is implemented by imposing the supplementary relations

$$[\sigma_a, \sigma_b] = \frac{1}{2} g_{ab} . \tag{40}$$

It is known that the centre is trivial in that case. Hence, the set of operators anticommuting with all the  $\sigma_a$  is  $\{0\}$ . In particular, the Scasimir operator vanishes. Moreover, it is easy to compute  $A_{2k}^{(n)}$  and one finds  $A_{2k}^{(n)} = 2^{-k} \frac{n!}{(n-k)!}$ . This implies the following sum rule among the coefficients  $x_k^{(n)}$ :

$$\sum_{k=0}^n 2^{k-n} \frac{n!}{k!} x_k^{(n)} = 0 \tag{41}$$

which amounts, using (38), to

$$\sum_{k=0}^n \binom{2n}{2k} \frac{d^{2k}}{du^{2k}} \left( \frac{u}{\sinh u} \right)^{2n+1} \Big|_{u=0} = e^{-u} \frac{d^{2n}}{du^{2n}} e^u \left( \frac{u}{\sinh u} \right)^{2n+1} \Big|_{u=0} = 0 . \tag{42}$$

This formula can be proved directly using Cauchy's residue theorem. Indeed, choosing a small contour around the origin, the last expression becomes

$$\oint \frac{du}{u^{2n+1}} e^u \left( \frac{u}{\sinh u} \right)^{2n+1} = 0 \quad \text{for } n \geq 1 . \tag{43}$$

Using the variable  $t = \sinh u$ , this follows from the fact that the function  $\frac{e^u}{\cosh u} - 1$  is an odd function of  $t$ .

### 5.3 On the square of the Scasimir

The square of the Scasimir is obviously a Casimir element. We shall now characterize this Casimir operator. It is known that Harish-Chandra's theorem holds for  $osp(1|2n)$  [8], i.e. there exists an isomorphism between the space of Weyl invariant polynomials in the Cartan generators and the centre  $\mathcal{Z}$  of  $\mathcal{U}(osp(1|2n))$ . Using a Poincaré–Birkhoff–Witt basis, any element  $x$  can be written  $x = x_0 + x_1$  with  $x_0 \in \mathcal{U}(\mathcal{H})$  and  $x_1 \in \mathcal{N}^- \mathcal{U}(osp(1|2n)) + \mathcal{U}(osp(1|2n)) \mathcal{N}^+$  where  $\mathcal{N}^- \oplus \mathcal{H} \oplus \mathcal{N}^+$  is the standard Borel decomposition of  $osp(1|2n)$ . Let  $h$  be the projection  $x \mapsto x_0$  within this direct sum and  $\bar{h}$  its restriction to the centre  $\mathcal{Z}$ . The Harish-Chandra isomorphism is expressed as  $\gamma^{-1} \circ \bar{h}$  where  $\gamma$  is the automorphism of the ring of the polynomials in the Cartan generators defined by  $\gamma : H_\alpha \mapsto H_\alpha - (n - \alpha + \frac{1}{2})$ .

Using the explicit expression of the Scasimir, we deduce that  $\gamma^{-1} \circ \bar{h}(Sc^2)$  is a polynomial in the Cartan generators of degree  $2n$ , the power of each being at most equal to two since each  $\sigma_a$  appears only once in  $Sc$ . Moreover, this polynomial being symmetric, one has

$$\gamma^{-1} \circ \bar{h}(Sc^2) = \prod_{\alpha=1}^n H_\alpha^2. \quad (44)$$

The Casimir operator  $Sc^2$  is then equal to  $\bar{h}^{-1} \circ \gamma(\prod_{\alpha=1}^n H_\alpha^2)$ .

As a buy-product, one also has

$$\gamma^{-1} \circ h(Sc) = \prod_{\alpha=1}^n H_\alpha. \quad (45)$$

This last relation appears to have a direct generalization to the quantum case, whereas the generalization of (30) is much less obvious.

**Note added:** We thank Prof. I. Musson for sending us his preprints on superalgebras. One of them, “On the center of the enveloping algebra of a classical simple Lie superalgebra” (to appear in J. Algebra) contains, among other results, the proof of existence of the Scasimir for  $osp(1|2n)$  and a formula equivalent to (45).

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